Near Optimal Single-Track Gray Codes

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***Abstract*—Single-track Gray codes are a special class of Gray codes which have advantages over conventional Gray codes in certain quantization and coding applications. The problem of constructing high period single-track Gray codes is considered. Three iterative constructions are given, along with a heuristic method for obtaining good seed-codes. In combination, these yield many families of very high period single-track Gray codes. In particular, for $m \geq 3$, length $n = 2^m$, period $2^n - 2n$ codes are obtained.**

***Index Terms*—Gray codes, heuristic methods, inequivalent sequences, necklaces, quantization, recursive construction, self-dual sequences**

I. Introduction

A LENGTH n Gray code $\mathcal{C}$ is an ordered list of distinct binary n-tuples (called the codewords)

$$W_0, W_1, \cdots, W_{P-1}$$

having the property that any two adjacent codewords W_i and W_{i+1} differ in exactly one component. If this property holds for W_{P-1} and W_0 as well, we say the Gray code is *cyclic* with *period* the number of different codewords P. Otherwise, we say the Gray code is *acyclic*.

Constructions for Gray codes can be found in [1]–[3], while Gray codes have found application in diverse areas including coding theory [4], [5] and in the design of combinatorial algorithms [6], [7]. Another common use of Gray codes is in reducing quantization errors in various types of analog-to-digital conversion systems [1], [8]. As a typical example, a length n, period P Gray code can be used to record the absolute angular positions of a rotating wheel by encoding (e.g., optically) the codewords on n concentrically arranged tracks. n reading heads, mounted in parallel across the tracks suffice to recover the codewords. When the heads are nearly aligned with the division between two codewords, any components which change between those words will be in doubt and a spurious position value may result. Such quantization errors are minimized by using a Gray encoding, for then exactly one component can be in doubt and the two codewords that could possibly result identify the positions bordering the division, resulting in a small angular error.

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When high resolution is required, the need for a large number of concentric tracks results in encoders with large physical dimensions. This poses a problem in the design of small-scale or high-speed devices. Single-track Gray codes were proposed in [9] as a way of overcoming these problems. Let $\mathcal{C}$ be a length n cyclic Gray code with codewords $W_0, W_1, \cdots, W_{P-1}$ and write $W_i = [w_i^0, w_i^1, \cdots, w_i^{n-1}]$, so that w_i^j denotes component j of codeword i. We call the sequence

$$w_0^j, w_1^j, \cdots, w_{P-1}^j$$

of period P *component sequence* j of $\mathcal{C}$.

Definition 1: Let $\mathcal{C}$ be a length n, period P cyclic Gray code. Suppose that for each $1 \leq j < n$, there exists k_j with $0 \leq k_j < P$ such that component sequence j is a cyclic shift by k_j of component sequence 0, i.e.

$$w_0^j, w_1^j, \cdots, w_{P-1}^j = w_{k_j}^0, w_{k_j+1}^0, \cdots, w_{k_j+P-1}^0$$

(where subscripts are reduced modulo P). Then we say that $\mathcal{C}$ is a single-track Gray code.

As an immediate consequence, in a single-track Gray code W_i is actually equal to

$$[w_i^0, w_{i+k_1}^0, w_{i+k_2}^0, \cdots, w_{i+k_{n-1}}^0]$$

so that all the components of W_i can be obtained from component sequence 0. Thus in the quantization application above, the bits of any codeword can be obtained solely from a single track corresponding to component sequence 0, if the n reading heads are spaced around that single track at fixed relative positions $0, k_1, k_2, \cdots, k_{n-1}$. If a suitable single-track Gray code is available, then an encoder can be made significantly smaller in size.

The following necessary conditions on the parameters n and P of a single-track Gray code were established in [9]:

Lemma 2: Suppose there exists a length n, period P single-track Gray code $\mathcal{C}$. Then P is an even multiple of n and

$$2n \leq P \leq 2^n.$$

Examples of high period codes for small n were given in [9], along with a general construction for single-track Gray codes which leads to the following:

Result 3 ([9], Theorem 12): Suppose $n \geq 4$. Then there exists a length n, period nt single-track Gray code for every even t satisfying

$$2 \leq t \leq 2^{n-\left\lceil\sqrt{2(n-3)}\right\rceil-1}.$$

This result makes available codes with a large range of parameters. However, in an application, a code of a particular period using the smallest possible number of reading heads n

is required. The codes from Result 1 do not generally offer this property, although the result does allow us to obtain codes of most periods with reasonable efficiency. It is also of interest to determine, for each n, the highest possible period of a length n single-track Gray code. It is this question we address in this paper, though the constructions we give for high-period codes are usually easily adapted to produce codes of a particular period.

In Section II we review some background concepts and give a key method in the construction of our codes. The method is then applied to give a construction which produces length $2n$ codes from length n codes satisfying certain additional constraints. The construction can be iterated and beginning with some "seed-codes" we obtain several families of high-period single-track Gray codes. In Section IV we generalize our techniques to give a construction which builds a length $(k+1)n$ code from a length n code having additional properties, for each $k \geq 2$. Again, our method can be iterated to produce families of good codes. Section V contains a construction method for codes based on self-dual words. The method is applied to produce length n, period $2^n - 2n$ single-track Gray code for n a power of 2. From Lemma 1, these codes are near-optimal. In Section VI, we present a heuristic method which produces some optimal codes and some seeds for our constructions. We end with some conclusions and open problems.

II. A Basic Code Construction

We begin by introducing a number of concepts that will be useful in the constructions of the following sections.

We number the positions in a binary n-tuple (or *word*)

$$X = [x_0, x_1, \cdots, x_{n-1}]$$

from left to right by 0 to $n-1$. If W_i and W_{i+1} are adjacent words in a Gray code, then diff(W_i, W_{i+1}) will denote the unique position in which they differ. 0^m denotes a string of m zeros and 1^m denotes a string of m ones. The left-shift operator E acting on n-tuple X is defined by

$$E[x_0, x_1, \cdots, x_{n-1}] = [x_1, \cdots, x_{n-1}, x_0].$$

We will say that two n-tuples X, Y are (cyclically) *equivalent* if $E^t X = Y$ for some t. Otherwise, they are said to be *inequivalent*. The set of words in an equivalence class under this relation is called a *necklace*, and we will represent a necklace by any of its words. The period of an n-tuple X is defined to be the least positive t such that $E^t X = X$. We say that X (and the necklace containing X) is *full-period* if it has period n.

Next we give a construction for single-track Gray codes based on a kind of Gray code for necklaces.

Theorem 4: Let $S_0, S_1, \cdots, S_{r-1}$ be r inequivalent full-period n-tuples and suppose that for $0 \leq i < r-1$, S_i and S_{i+1} differ in exactly one position and that for some l relatively prime to n, $E^l S_0$ differs from S_{r-1} also in exactly one position. Then the words

$$\begin{array}{llll} S_0, & S_1, & \cdots & S_{r-1}, \\ E^l S_0, & E^l S_1, & \cdots & E^l S_{r-1}, \\ E^{2l} S_0, & E^{2l} S_1, & \cdots & E^{2l} S_{r-1}, \\ & \vdots & \vdots & \\ E^{(n-1)l} S_0, & E^{(n-1)l} S_1, & \cdots & E^{(n-1)l} S_{r-1} \end{array}$$

constitute a single-track Gray code of length n and period nr.

The special case $l = 1$ of the above theorem is due to Brandestini.

Proof: Since l is relatively prime to n, the integers $0, l, 2l, \cdots, (n-1)l$ are distinct modulo n. It is then clear from the properties of the words $S_0, S_1, \cdots, S_{r-1}$ that the list of words in the statement of the theorem do form a cyclic Gray code. We need only show that this code has the single-track property. Suppose the words $S_0, S_1, \cdots, S_{r-1}$ are written in a vertical list to form an $r \times n$ binary array. Let $C_0, C_1, \cdots, C_{n-1}$ be the columns of this array. Then it is easy to see that component sequence j of the code is $C_j, C_{j+l}, C_{j+2l}, \cdots, C_{j+(n-1)l}$ (with subscripts modulo n), formed by concatenation of the columns. In particular, component sequence 0 is just $C_0, C_l, C_{2l}, \cdots, C_{(n-1)l}$ and contains all the columns in some order. Now, since l and n are relatively prime, for every j we have $j = t_j l \bmod n$ for some t_j. Then component sequence j is the sequence $C_{t_j l}, C_{t_j l+l}, C_{t_j l+2l}, \cdots, C_{t_j l+(n-1)l}$, which is simply the shift by $t_j l r = jr$ of component sequence 0. Hence the code is single-track. □

Example 1: The list of words $[00001], [00011], [10011], [11011], [11010], [10010]$, satisfies the hypotheses of Theorem 1 and lead to a length 5, period 30 single-track Gray code with component sequence 0 equal to

$$0,0,1,1,1,1,0,0,0,1,1,0,0,0,0,0,0,0,0, \quad 1,1,1,1,1,1,1,1,1,0,0.$$

This is the length 5 code presented in [9].

III. A Construction Based on Necklaces

We will give a construction which begins with a Gray code on inequivalent words of period n, with certain properties, and produces a Gray code on inequivalent words of length $2n$, with the same properties. By iterating the construction on seed-codes and using the codes which result in Theorem 1, we will obtain families of single-track Gray codes of length $2^k n$ with high periods.

Let $\mathcal{X}(n)$ denote the set of 2^{n-1} words of the form

$$[x_0, x_1, \cdots, x_{n-2}, 0].$$

Lemma 5: Suppose $\mathcal{S} = \{S_0, S_1, \cdots, S_{r-1}\}$ is a set of r inequivalent full-period n-tuples, each one ending with a one. Then the set

$$\{[X, X + S_i] : 0 \leq i < r, X \in \mathcal{X}(n)\}$$

consists of $2^{n-1} r$ inequivalent words of period $2n$ having a zero in position $n-1$ and a one in position $2n-1$.

Proof: Assume $X, X' \in \mathcal{X}(n)$ and $[X, X+S_i]$ is equivalent to $[X', X'+S_j]$. So $E^c[X, X+S_i] = [X', X'+S_j]$ for some $0 \le c < 2n$. Now

$$(E^n+1)E^c[X, X+S_i] = [E^cS_i, E^cS_i]$$

while

$$(E^n+1)[X', X'+S_j] = [S_j, S_j].$$

But for $i \ne j$, S_i and S_j are inequivalent, so we must have $i = j$ and $S_i = E^cS_i$. But S_i is a full-period word, so we can deduce that either $c = 0$ or $c = n$. If $c = 0$ we see that $X = X'$ also, so we must have $c = n$. But then we have $[X+S_i, X] = [X', X'+S_i]$ which is impossible since $X+S_i$ ends with a one and X' ends with a zero. The observation that we obtain $2^{n-1}r$ inequivalent period $2n$ words in which there is a zero in position $n-1$ and a one in position $2n-1$ follows immediately. □

The next lemma is immediate.

Lemma 6: If $S_0, S_1, \cdots, S_{r-1}$ is a cyclic Gray code then for $X \in \mathcal{X}(n)$

$$[X, X+S_0], [X, X+S_1], \cdots, [X, X+S_{r-1}]$$

is a cyclic Gray code.

The next lemma will be useful in merging one Gray code into another.

Lemma 7: If $X, X' \in \mathcal{X}(n)$ differ only in position d and $S_0, S_1, \cdots, S_{r-1}$ is a cyclic Gray code in which S_i and S_{i+1} differ in position d then

$$\begin{gathered}[X, X+S_i],\\ [X', X'+S_{i+1}],\\ \vdots\\ [X', X'+S_{r-1}],\\ [X', X'+S_0],\\ \vdots\\ [X', X'+S_i],\\ [X, X+S_{i+1}]\end{gathered}$$

is a cyclic Gray code.

Proof: If S_i and S_{i+1} differ in position d and X and X' differ in position d then $[X, X+S_i]$ and $[X', X'+S_{i+1}]$ differ only in position d. Similarly, $[X', X'+S_i]$ and $[X, X+S_{i+1}]$ differ only in position d. □

Construction A

Assume $S_0, S_1, \cdots, S_{r-1}$ are r inequivalent full-period n-tuples, $n \ge 7$, with the following four properties:

A1. For each i, S_i and S_{i+1} (subscripts taken modulo r) differ in exactly one position.

A2. Let $\mathcal{D}_n = \{\text{diff}(S_i, S_{i+1}) : 1 \le i < r-2\}$. Then

$$\mathcal{D}_n = \{0, 1, \cdots, n-2\}.$$

A3. Position $n-1$ in each S_i is a one.

A4. $S_{r-2} = [0^{n-3}1\,1\,1], S_{r-1} = [0^{n-3}0\,1\,1], S_0 = [0^{n-3}0\,0\,1], S_1 = [0^{n-3}1\,0\,1]$.

Example 2: For $n = 7$, the 18 full-period necklaces can be ordered as follows to satisfy these properties:

$$\begin{array}{ll} S_0 = [0000001] & S_9 = [0110101] \\ S_1 = [0000101] & S_{10} = [0110111] \\ S_2 = [0001101] & S_{11} = [0100111] \\ S_3 = [0001001] & S_{12} = [0100101] \\ S_4 = [1001001] & S_{13} = [1100101] \\ S_5 = [1011001] & S_{14} = [1000101] \\ S_6 = [1111001] & S_{15} = [1000111] \\ S_7 = [1111101] & S_{16} = [0000111] \\ S_8 = [0111101] & S_{17} = [0000011] \end{array}$$

We list the elements of $\mathcal{X}(n)$ in an order

$$X_0, X_1, \cdots, X_{2^{n-1}-1}$$

such that for each $i \ge 1$, X_i differs in exactly one position from some word appearing earlier in the list, i.e., from some X_j with $j < i$. We assume that $X_0 = [0^n]$. We proceed by generating the 2^{n-1} cyclic Gray codes guaranteed by Lemmas 5 and 6. We label the code corresponding to word X_i by $\mathcal{X}_i$. We will merge these 2^{n-1} cyclic Gray codes one by one into a main code using Lemma 7. We take as the initial main code the cyclic Gray code $\mathcal{X}_0$. Assume that the codes $\mathcal{X}_1, \mathcal{X}_2, \cdots, \mathcal{X}_{l-1}$ have been successively inserted into the main code. We will show that $\mathcal{X}_l$ can also be inserted into the main code.

Now there exists a word X_j with $j < l$ such that X_j and X_l differ in exactly one position, d say, and there exists a pair of words S_i and S_{i+1} that also differ in exactly position d, where $1 \le i < r-2$. We claim that the words $[X_j, X_j+S_i]$ and $[X_j, X_j+S_{i+1}]$ still lie adjacent in the main code. For if not, then some code $\mathcal{X}_m$ $(m \ne l)$ must have been inserted between them. This only occurs if X_j and X_m differ in exactly position d. This in turn implies that $X_l = X_m$—a contradiction, since these words are distinct. Therefore, we can insert a cyclic shift of the listing of the words of the code $\mathcal{X}_l$ between the words $[X_j, X_j+S_i]$ and $[X_j, X_j+S_{i+1}]$ using Lemma 7, extending the main code.

Executing this process beginning with $\mathcal{X}_1$ and ending with $\mathcal{X}_{2^{n-1}-1}$, we finally obtain a cyclic Gray code of period $2^{n-1}r$.

Observe that in the above procedure, we never insert the words of $\mathcal{X}_i$ between words

$$[X_j, X_j+S_{r-2}], [X_j, X_j+S_{r-1}], [X_j, X_j+S_0], [X_j, X_j+S_1]$$

of the earlier code $\mathcal{X}_j$. Thus for each $X \in \mathcal{X}(n)$, the words

$$[X, X+S_{r-2}], [X, X+S_{r-1}], [X, X+S_0], [X, X+S_1]$$

will always be consecutive in the final code. In particular, we see that the final two and first two words in the code are just

$$\begin{aligned} [0^n, 0^n+S_{r-2}] &= [0^{2n-3}111] \\ [0^n, 0^n+S_{r-1}] &= [0^{2n-3}011] \\ [0^n, 0^n+S_0] &= [0^{2n-3}001] \\ [0^n, 0^n+S_1] &= [0^{2n-3}101] \end{aligned}$$

so that the new code has property A4. Properties A1 and A3 are satisfied as an immediate consequence of the construction and Lemmas 5, 6, and 7. Clearly from Lemma 7 and the construction, we have $\mathcal{D}_{2n} = \{0, 1, \cdots, n-2, n, n+1, \cdots, 2n-2\}$.

To satisfy Property A2, we need to modify the code so that $\mathcal{D}_{2n}$ also contains $n-1$. Taking $X = [110^{n-2}]$, we use the above observation again to see that the final code will have as four consecutive words

$$\begin{aligned}[X, X+S_{r-2}] &= [110^{n-2}110^{n-5}111]\\ [X, X+S_{r-1}] &= [110^{n-2}110^{n-5}011]\\ [X, X+S_0] &= [110^{n-2}110^{n-5}001]\\ [X, X+S_1] &= [110^{n-2}110^{n-5}101].\end{aligned}$$

On the other hand, taking $X' = [10^{n-1}]$, we obtain as consecutive words

$$\begin{aligned}[X', X'+S_{r-2}] &= [10^{n-2}010^{n-4}111]\\ [X', X'+S_{r-1}] &= [10^{n-2}010^{n-4}011].\end{aligned}$$

Now $[X, X+S_{r-2}]$ and $[X, X+S_1]$ differ only in position $2n-2$ while the sequence of words

$$\begin{aligned}[X', X'+S_{r-2}] &= [10^{n-2}010^{n-4}111]\\ E[X, X+S_{r-1}] &= [10^{n-2}110^{n-4}111]\\ E[X, X+S_0] &= [10^{n-2}110^{n-4}011]\\ [X', X'+S_{r-1}] &= [10^{n-2}010^{n-4}011]\end{aligned}$$

form a Gray code. Thus we can remove words $[X, X+S_{r-1}]$ and $[X, X+S_0]$ from the code and reinsert their shifts between $[X', X'+S_{r-2}]$ and $[X', X'+S_{r-1}]$, maintaining Property A3, to obtain a cyclic Gray code of inequivalent words with $\mathcal{D}_{2n} = \{0, 1, \cdots, \cdots, 2n-2\}$, i.e., with Property A2.

Then we have:

Theorem 8: Suppose there exists a cyclic Gray code of length n, $n \geq 7$ and period r with Properties A1 to A4. Then there exists a cyclic Gray code of length $2n$ and period $2^{n-1}r$ with the same properties.

Beginning with the "seed-code" of Example 2 and iterating the use of Construction A we obtain, for each $k \geq 1$, a length $7 \cdot 2^k$, period $18 \cdot 2^{7(2^k-1)-k}$ Gray code consisting of inequivalent words with first word $[0^{7\cdot 2^k-1}1]$ and last word $[0^{7\cdot 2^k-2}11]$. Notice that $E[0^{7\cdot 2^k-1}1]$ and $[0^{7\cdot 2^k-2}11]$ differ in exactly one position. We can therefore apply Theorem 4 to obtain a family of single-track Gray codes of length $7\cdot 2^k$ and period $18 \cdot 7 \cdot 2^{7(2^k-1)} = \frac{63}{64} \cdot 2^{7\cdot 2^k}$. This period is a very high (and constant) fraction of the upper bound from Lemma 2.

We will show in Section VI that there also exist seed-codes with 56 full-period necklaces for $n = 9$ and 96 full-period necklaces for $n = 10$. Using these in the same way as the length 7 seed above, we can construct a family of length $9\cdot 2^k$, period $\frac{63}{64}\cdot 2^{9\cdot 2^k}$ single-track Gray codes and a family of length $10 \cdot 2^k$, period $\frac{15}{16} \cdot 2^{10\cdot 2^k}$ single-track Gray codes.

IV. An Extended Construction

In this section we extend the technique of Construction A to give a construction method which begins with a Gray code on inequivalent words of period n, with certain properties, and produces a Gray code on inequivalent words of length $(k+1)n$, with the same properties for $k \geq 2$. By iterating the construction on seed-codes and using the codes which result in Theorem 4, we will obtain families of single-track Gray codes of lengths a multiple of n with high periods.

We begin with a suitable analog of Lemma 5. Suppose that $k \geq 2$ and that $\mathcal{X}$ and $\mathcal{Y}$ are disjoint sets of n-tuples. We write $|\mathcal{X}| = s$ and $|\mathcal{Y}| = t$.

Lemma 9: Let $\mathcal{S} = \{S_0, S_1, \cdots, S_{r-1}\}$ be a set of r inequivalent full-period n-tuples. Then the set

$$\{[X, Y_0, \cdots, Y_{k-2}, X+Y_0+\cdots+Y_{k-2}+S_i] : 0 \leq i < r,\ X \in \mathcal{X}, Y_j \in \mathcal{Y}\}$$

consists of $r \cdot s \cdot t^{k-1}$ inequivalent full-period $(k+1)n$ -tuples.

Proof: Assume that $X, X' \in \mathcal{X}$, $Y_i, Y_i' \in \mathcal{Y}$ $(0 \leq i \leq k-2)$ and that

$$E^c[X, Y_0, \cdots, Y_{k-2}, X+Y_0+\cdots+Y_{k-2}+S_i] = [X', Y_0', \cdots, Y_{k-2}', X'+Y_0'+\cdots+Y_{k-2}'+S_j]$$

for some $0 \leq c < kn$. Then, similarly as in the proof of Lemma 5, we apply the operator $E^{kn}+\cdots+E^n+1$ to each of these words to see that $E^cS_i = S_j$. Since the words of $\mathcal{S}$ are inequivalent and have period n, we see that $i = j$ and $c = 0 \bmod n$. We write $c = ln$ for some $0 \leq l \leq k$. If $l = k$, we see immediately that $X = Y_0'$, a contradiction since the sets $\mathcal{X}$ and $\mathcal{Y}$ are disjoint. If $1 \leq l < k$ then we have $X' = Y_{l-1}$, again contradicting the disjointness. So we must have $l = 0$, and hence $c = 0$. Thus $X = X'$ and $Y_i = Y_i'$ for each i, and the Lemma follows immediately. □

The next lemma is immediate.

Lemma 10: If $S_0, S_1, \cdots, S_{r-1}$ is a cyclic Gray code then for $X \in \mathcal{X}$ and $Y_i \in \mathcal{Y}$ $(0 \leq i \leq k-2)$, the list of words

$$\begin{gathered}[X, Y_0, \cdots, Y_{k-2}, X+Y_0+\cdots+Y_{k-2}+S_0]\\ [X, Y_0, \cdots, Y_{k-2}, X+Y_0+\cdots+Y_{k-2}+S_1]\\ \vdots\\ [X, Y_0, \cdots, Y_{k-2}, X+Y_0+\cdots+Y_{k-2}+S_{r-1}]\end{gathered}$$

is a cyclic Gray code.

Now we give a lemma which will be useful in merging Gray codes in the constructions to follow. The proof is similar to that of Lemma 7.

Lemma 11: Suppose that for some $0 \leq j \leq k-2$, $Y_j, Y_j' \in \mathcal{Y}$ differ only in position d and $X \in \mathcal{X}$ and $Y_0, \cdots, Y_{j-1}, Y_{j+1}, \cdots, Y_{k-2} \in \mathcal{Y}$ are arbitrary. Let $S_0, S_1, \cdots, S_{r-1}$ be a cyclic Gray code in which S_i and S_{i+1} differ in position d. Then the list at the top of the following page is a Gray code in which the first and last pairs of words differ in position $d+(j+1)n$.

Of course, a similar result to that above holds for pairs $X, X' \in \mathcal{X}$ which differ in exactly one position.

Construction B

Assume $S_0, S_1, \cdots, S_{r-1}$ are r inequivalent full-period n-tuples, $n \geq 7$, with the following properties:

- B1. For each i, S_i and S_{i+1} (subscripts taken modulo r) differ in exactly one position.
- B2. Let $\mathcal{D}_n = \{\text{diff}(S_i, S_{i+1}) : 0 \leq i < r-1\}$. Then
$$\mathcal{D}_n = \{0, 1, \cdots, n-1\}.$$
- B3. $S_0 = [10^{n-1}]$, $S_{r-1} = [10^{n-2}1]$.

$$\begin{array}{l}[X, Y_0, \cdots, Y_j, \cdots, Y_{k-2}, X + Y_0 + \cdots + Y_j + \cdots + Y_{k-2} + S_i] \\ [X, Y_0, \cdots, Y_j', \cdots, Y_{k-2}, X + Y_0 + \cdots + Y_j' + \cdots + Y_{k-2} + S_{i+1}] \\ \vdots \\ [X, Y_0, \cdots, Y_j', \cdots, Y_{k-2}, X + Y_0 + \cdots + Y_j' + \cdots + Y_{k-2} + S_{r-1}] \\ [X, Y_0, \cdots, Y_j', \cdots, Y_{k-2}, X + Y_0 + \cdots + Y_j' + \cdots + Y_{k-2} + S_0] \\ \vdots \\ [X, Y_0, \cdots, Y_j', \cdots, Y_{k-2}, X + Y_0 + \cdots + Y_j' + \cdots + Y_{k-2} + S_i] \\ [X, Y_0, \cdots, Y_j, \cdots, Y_{k-2}, X + Y_0 + \cdots + Y_j + \cdots + Y_{k-2} + S_{i+1}]\end{array}$$

Suppose further that $\mathcal{X} = X_0, X_1, \cdots, X_{s-1}$ and $\mathcal{Y} = Y_0, Y_1, \cdots, Y_{t-1}$ are a pair of disjoint Gray codes of length n with the following properties:

B4. Let

$$\mathcal{E}_n = \{\text{diff}(X_i, X_{i+1}) : 0 \le i < s-1\}$$
$$\mathcal{F}_n = \{\text{diff}(Y_i, Y_{i+1}) : 0 \le i < t-1\}.$$

Then

$$\mathcal{E}_n = \mathcal{F}_n = \{0, 1, \cdots, n-1\}.$$

B5. $X_0 = [10^{n-1}]$ and $Y_0 = [0^n]$.

Then we show that all the distinct length $(k+1)n$ necklaces resulting from Lemma 9 can be arranged to form a Gray code satisfying Properties B1 to B3. We will go on to show how to construct pairs of codes $\mathcal{X}$ and $\mathcal{Y}$ simultaneously having Properties B4 and B5 and maximizing the period of the single-track codes resulting from Theorem 4.

For each $X \in \mathcal{X}$ and $(k-1)$-tuple

$$(W_0, W_1, \cdots, W_{k-2}) \in \mathcal{Y}^{k-1}$$

we form the length $(k+1)n$ cyclic Gray code

$$\mathcal{C}(X, W_0, W_1, \cdots, W_{k-2})$$

with words

$$[X, W_0, \cdots, W_{k-2}, X + W_0 + \cdots + W_{k-2} + S_i] \quad (0 \le i < r)$$

using Lemma 10. By Lemma 9, the $s \cdot t^{k-1}$ codes of period r obtained in this way are disjoint and consist of inequivalent full-period words. We will merge these codes into a main code using Lemma 11 in a way that is similar to that used in Construction A. Initially, we take our main code to be $\mathcal{C}(X_0, Y_0, Y_0, \cdots, Y_0)$, whose first and last words are

$$[X_0, Y_0, \cdots, Y_0, X_0 + Y_0 + \cdots + Y_0 + S_0] = [10^{(k+1)n-2}0]$$
$$[X_0, Y_0, \cdots, Y_0, X_0 + Y_0 + \cdots + Y_0 + S_{r-1}] = [10^{(k+1)n-2}1].$$

Now Y_0 and Y_1 differ in exactly one position, d say, and by Property B2 and Lemma 11 we can insert a cyclic shift of code $\mathcal{C}(X_0, Y_0, \cdots, Y_0, Y_1)$ into the main code, without disturbing its first and last codewords. The inserted words constitute a list which, from Property B2, contains adjacent words differing in exactly position $nk + d$ for each $0 \le d < n$. Using arguments similar to those in Section II, each of the codes $\mathcal{C}(X_0, Y_0, \cdots, Y_0, Y_i)$ can be successively inserted into the main code, the last insertion being of code $\mathcal{C}(X_0, Y_0, \cdots, Y_0, Y_{t-1})$. Now since Y_0 and Y_1 differ in exactly one position, we can insert $\mathcal{C}(X_0, Y_0, \cdots, Y_0, Y_1, Y_{t-1})$ followed by codes

$$\mathcal{C}(X_0, Y_0, \cdots, Y_0, Y_1, Y_{t-2}), \cdots, \mathcal{C}(X_0, Y_0, \cdots, Y_0, Y_1, Y_0)$$

into the main code. Then we continue by inserting code $\mathcal{C}(X_0, Y_0, \cdots, Y_0, Y_2, Y_0)$, etc. It should be clear how this process can be continued to insert all the codes $\mathcal{C}(X, W_0, W_1, \cdots, W_{k-2})$ into the main code, while leaving undisturbed the first and last codewords. Notice that our choice of insertion order for the codes corresponds to the ordering of the words of $\boldsymbol{Z}_s \times \boldsymbol{Z}_t^{k-1}$ as a Gray code.

The final code of $r \cdot s \cdot t^{k-1}$ inequivalent full-period $(k+1)n$-tuples clearly satisfies Properties B1 and B3. We need to verify Property B2. This is a simple consequence of the remarks about the first and last pairs of words in Lemma 11 and Property B4 of the codes $\mathcal{X}$ and $\mathcal{Y}$.

In order to apply this construction, we need to prove the existence of pairs of Gray codes having Properties B4 and B5. We also wish to maximize the number of necklaces in the final code. Clearly, we should take $s + t = 2^n$ to include all length n words in the sets $\mathcal{X}$ and $\mathcal{Y}$. In this case, we have:

Theorem 12: Suppose $n \ge 3$ and $n+1 \le s \le 2^{n-1}$. Then there exists a pair of disjoint length n Gray codes $\mathcal{X}$ and $\mathcal{Y}$ with $|\mathcal{X}| = s$ and $|\mathcal{Y}| = 2^n - s$ which satisfy Properties B4 and B5.

To prove the Theorem, we need the following:

Construction C

Let $\mathcal{C}_2$ be the length 2 cyclic Gray code with words

$$[11], [10], [00], [01].$$

For $n \ge 3$, we recursively define a length n, period 2^n cyclic Gray code $\mathcal{C}_n$ as follows: let the words of $\mathcal{C}_{n-1}$ be

$$W_0, W_1, \cdots, W_{2^{n-1}-1}.$$

Then we take as the words of $\mathcal{C}_n$ the list

$$\begin{gathered}[W_0,1]\\ [W_0,0]\\ [W_1,0]\\ \vdots\\ [W_{2^{n-1}-1},0]\\ [W_{2^{n-1}-1},1]\\ \vdots\\ [W_1,1].\end{gathered}$$

A simple inductive argument shows that $\mathcal{C}_n$, a variant of the standard binary-reflected Gray code, has the property that its first $n+2$ words are

$$\begin{aligned} W_0 &= [1^n]\\ W_1 &= [1^{n-1}0]\\ W_2 &= [1^{n-2}0^2]\\ &\vdots\\ W_{n-1} &= [10^{n-1}]\\ W_n &= [0^n]\\ W_{n+1} &= [010^{n-2}].\end{aligned}$$

Proof: (of Theorem 12) Suppose $n+1 \le s \le 2^{n-1}$ and let $\mathcal{C}_n = W_0, W_1, \cdots, W_{2^n-1}$ be constructed according to Construction C. The set of words $\mathcal{C}'_n = X_0, X_1, \cdots, X_{2^n-1}$ with

$$X_i = EW_i + [10^{n-1}]$$

also form a cyclic Gray code with $X_n = [10^{n-1}]$ and $X_{n+1} = [0^n]$. We take

$$\mathcal{X} = X_n, X_{n-1}, \cdots, X_0, X_{2^n-1}, \cdots, X_{2^n+n-s+1}$$

and

$$\mathcal{Y} = X_{n+1}, X_{n+2}, \cdots, X_{2^n+n-s}.$$

Clearly $\mathcal{X}$ and $\mathcal{Y}$ are disjoint Gray codes while $\mathcal{X}$ contains s words and begins with $[10^{n-1}]$ and $\mathcal{Y}$ contains $2^n - s$ words and begins with $[0^n]$. We need only check that Property B4 holds for $\mathcal{X}$ and $\mathcal{Y}$. This is clear for $\mathcal{X}$ from its construction from $\mathcal{C}_n$ and then follows automatically for $\mathcal{Y}$ from the fact that $\mathcal{Y}$ contains at least half of all words of length n, those words being the words not in $\mathcal{X}$. □

Combining Construction B with Theorem 12, we have:

Theorem 13: Suppose there exists an arrangement of r inequivalent full-period n-tuples, $n \ge 7$, satisfying Properties B1, B2, and B3. Suppose further that

$$n+1 \le s \le 2^{n-1}.$$

Then there also exists an arrangement of

$$r \cdot s \cdot (2^n - s)^{k-1}$$

inequivalent full-period $(k+1)n$-tuples satisfying the same properties.

It is a simple exercise to show that either the choice

$$s = \left\lfloor \frac{1}{k} 2^n \right\rfloor \quad \text{or} \quad s = \left\lceil \frac{1}{k} 2^n \right\rceil$$

maximizes the number of necklaces $r \cdot s \cdot (2^n - s)^{k-1}$ in the above theorem. In order that the conditions of Theorem 12 be satisfied, however, we must have $n+1 \le s \le 2^{n-1}$ which leads to the restriction that

$$2 \le k \le \left\lfloor \frac{2^n}{n+1} \right\rfloor.$$

The number of necklaces obtained from the above choices of s are then roughly

$$r \cdot \frac{(k-1)^{k-1}}{k^k} \cdot 2^{nk}.$$

Notice that by virtue of Property B3, the codes in Theorem 13 can be used immediately in Theorem 4 to produce length $(k+1)n$ single-track Gray codes of period roughly

$$(k+1)n \cdot r \cdot \frac{(k-1)^{k-1}}{k^k} \cdot 2^{nk}.$$

We will give a method for obtaining seed-codes in Section VI, but for now we illustrate the power of our construction with an example.

Example 3: For $n = 7$, there exists an arrangement of all 18 full-period necklaces satisfying Properties B1 to B3: such a code can be obtained from the code of Example 2 by applying E^2 to the word S_0 and E^1 to S_{17} to obtain a code satisfying Property B2, then reversing all the codewords and applying E^2 to each word to ensure Property B3.

Now $\lfloor \frac{2^n}{n+1} \rfloor = 16$ and so we can obtain codes of lengths $7(k+1)$ for $2 \le k \le 16$. All these codes will have Properties B1 to B3. Taking $k = 2$, we obtain a code of $18 \cdot 2^{12}$ length 21 necklaces, which on applying Theorem 4 gives a single-track Gray code of length 21 and period $\frac{189}{256} \cdot 2^{21}$, roughly three-quarters of the maximum period allowed by the necessary conditions. Taking $k = 8$, we will finally obtain a length 63, period $0.435 \cdot 2^{63}$ single-track Gray code.

Note that if we do not require a construction which can be iterated, then Property B2 is not needed for the final code. In this case, we can relax the condition on s in Theorem 13: we need only that $\mathcal{X}$ and $\mathcal{Y}$ be disjoint Gray codes containing the words $[10^{n-1}]$ and $[10^{n-2}1]$, respectively (so that Theorem 4 can be easily applied). Length $(k+1)n$ codes of period $(k+1)n \cdot rs(2^n - s)^{k-1}$ can then be obtained for every $k \ge 2$ and $1 \le s \le 2^n - 1$.

V. A Construction from Self-Dual Words

In this section, we will give a construction for length n codes based on self-dual words of length $2n$. This generalizes the examples for $n = 6$ and $n = 8$ given in [9].

A word S is *self-dual* if $S = [X, \overline{X}]$ for some X. Clearly, for any i, $E^i S = [Y, \overline{Y}]$ for some Y. Hence we have the following lemma.

Lemma 14: Let S_1 and S_2 be two inequivalent full-period self-dual $2n$-tuples. Then $2n$ distinct n-tuples appear as subsequences of consecutive bits in each of S_1 and S_2, while none of the n-tuples appearing in S_1 appear in S_2.

Lemma 14 leads to the following idea for constructing single-track Gray codes. Let $S_0, S_1, \cdots, S_{r-1}$ be r inequivalent full-period self-dual $2n$-tuples. We write $S_i = [X_i, \overline{X}_i]$. Assume further that for $0 \leq i < r-1$, S_i and S_{i+1} differ in exactly two positions (one between X_i and X_{i+1} and one between $\overline{X}_i$ and $\overline{X}_{i+1}$), and that for some j relatively prime to $2n$, $E^j S_0$ differs from S_{r-1} also in exactly two positions. Let $S_i = [s_i^0, s_i^1, \cdots, s_i^{2n-1}]$ and denote an n-tuple of consecutive bits of S_i by

$$F^j S_i = [s_i^j, s_i^{j+1}, \cdots, s_i^{j+n-1}]$$

(where the superscript is taken modulo $2n$). Then we have the following result, whose proof is analogous to that of Theorem 4.

Theorem 15: The words

$$\begin{array}{cccc} F^0 S_0, & F^0 S_1, & \cdots & F^0 S_{r-1}, \\ F^j S_0, & F^j S_1, & \cdots & F^j S_{r-1}, \\ F^{2j} S_0, & F^{2j} S_1, & \cdots & F^{2j} S_{r-1}, \\ \vdots & & & \\ F^{2nj} S_0, & F^{2nj} S_1, & \cdots & F^{2nj} S_{r-1} \end{array}$$

constitute a single-track Gray code of length n and period $2nr$.

We will now present a recursive construction based on Theorem 15. Let $S_0, S_1, \cdots, S_{r-1}$ be the set of all inequivalent full-period self-dual words of length $2n$ and let $\mathcal{Y}(n)$ denote the set of 2^{n-1} elements consisting of the $2^{n-1}-1$ words of the form $[1, y_1, \cdots, y_{n-1}]$, where at least one of the y_i's is a zero, together with the word $[0^n]$. For each $S = [X, \overline{X}]$ of length $2n$ and for every $Y \in \mathcal{Y}(n)$, let

$$S_Y = [Y, X+Y, \overline{Y}, X+\overline{Y}].$$

Similarly to the proof of Lemma 5 and the proof of [10, Lemma 1], we have the following lemma:

Lemma 16: The set of words

$$\bigcup_{i=0}^{r-1} \mathcal{S}_i(n)$$

where

$$\mathcal{S}_i(n) = \bigcup_{Y \in \mathcal{Y}(n)} (S_i)_Y$$

contains $2^{n-1}r$ inequivalent self-dual words of length $4n$.

We will continue with our construction, but restrict ourselves to n which are a power of 2 since, as will be proven later, our construction for these n is optimal by this method. A similar construction based on essentially the same idea can be given for other values of n.

If n is a power of 2, then there are $r = 2^n/2n$ inequivalent full-period self-dual $2n$-tuples and these contain all the n-tuples as subsequences. Assume that $S_0, S_1, \cdots, S_{r-1}$, the set of all inequivalent self-dual words of length $2n$, are arranged so that the following three properties hold:

C1. For each i, S_i, and S_{i+1} (subscripts taken modulo r) differ in exactly two positions k and $k+n$ (subscripts taken modulo $2n$).

C2. Let $\text{diff}^*(S_i, S_{i+1})$ denote the first position in which S_i and S_{i+1} differ and let

$$\mathcal{D}_n = \{\text{diff}^*(S_i, S_{i+1}) : 0 \leq i < r-2\}.$$

Then

$$\mathcal{D}_n = \{0, 1, \ldots, n-1\}.$$

C3. $E(S_{r-2})$ differs in exactly two positions from S_0. More precisely, we require

$$\begin{aligned} S_{r-2} &= [0^{n-4}10001^{n-4}0111] \\ S_{r-1} &= [0^{n-4}10011^{n-4}0110] \\ S_0 &= [0^{n-4}00011^{n-4}1110]. \end{aligned}$$

Example 4: For $n = 8$, the 16 self-dual words of length 16 are ordered below so that Properties C1 to C3 hold

$$\begin{array}{ll} S_0 = [0000000111111110] & S_8 = [1111000100001110] \\ S_1 = [1000000101111110] & S_9 = [1101000100101110] \\ S_2 = [1000001101111100] & S_{10} = [1101100100100110] \\ S_3 = [1100001100111100] & S_{11} = [0101100110100110] \\ S_4 = [1100011100111000] & S_{12} = [0101100010100111] \\ S_5 = [1101011100101000] & S_{13} = [0100100010110111] \\ S_6 = [1101010100101010] & S_{14} = [0000100011110111] \\ S_7 = [1111010100001010] & S_{15} = [0000100111110110] \end{array}$$

Lemma 17: For any $Y \in \mathcal{Y}(n)$ the list of words

$$\mathcal{S}(Y) = (S_0)_Y, (S_1)_Y, \cdots, (S_{r-1})_Y$$

satisfy Property C1.

Proof: If X_i and X_{i+1} differ in exactly one position then so do the words $[Y, X_i + Y]$ and $[Y, X_{i+1} + Y]$. □

Lemma 18: If Y and Y' differ in exactly position d and $\text{diff}^*(S_i, S_{i+1}) = d$ then the list of words

$$(S_i)_Y, (S_{i+1})_{Y'}, (S_{i+2})_{Y'}, \cdots, (S_{r-1})_{Y'}, \\ (S_0)_{Y'}, \cdots, (S_i)_{Y'}, (S_{i+1})_Y$$

satisfy Property C1 above. The first and last pairs of words differ only in positions d and $d+2n$, while for every $n \leq d' < 2n$, some pair of consecutive words in the list differ only in positions d' and $d'+2n$.

Proof: If $S_i = [X_i, \overline{X}_i]$ where X_i and X_{i+1} differ exactly in position d and Y and Y' also differ in exactly position d, then $X_i + Y = X_{i+1} + Y'$ and $[Y, X_i + Y]$ and $[Y', X_{i+1} + Y']$ differ exactly in position d. Similarly, $[Y', X_i + Y']$ and $[Y, X_{i+1} + Y]$ differ exactly in position d. The statement about positions n up to $2n-1$ follows from the construction of the words $(S_j)_{Y'}$ and Property C2 of $S_0, S_1, \cdots, S_{r-1}$. □

Lemma 19: If the set of self-dual words of length $2n = 2^{m+1}$ can be arranged so as to satisfy Properties C1 to C3, then so can the set of self-dual words of length $4n$.

Proof: We start by forming the list of words $\mathcal{S}(Y)$ for each $Y \in \mathcal{Y}(n)$, as in Lemma 17. Next, we merge these lists $\mathcal{S}(Y)$ using Lemma 18 in a similar way as we did in Construction A. We order the words of $\mathcal{Y}(n)$ as follows: we take $Y_0 = [0^n]$, $Y_j = [1^j 0^{n-j}]$, $1 \le j \le n-1$, and $Y_n = [10^{n-2}1]$. Then we order the remaining words of $\mathcal{Y}(n)$ so that each Y_i differs in exactly one position from some Y_j, $j < i$. Notice that for $1 \le j < n$, Y_j differs from Y_{j-1} in position $j-1$, while Y_n differs from Y_1 in position $n-1$.

We take as the initial main list $\mathcal{S}(Y_0)$. Assume that the lists $\mathcal{S}(Y_1), \mathcal{S}(Y_2), \cdots, \mathcal{S}(Y_{l-1})$ have been successively inserted into the main list. We will show that $\mathcal{S}(Y_l)$ can also be introduced.

Now there exists a word Y_j with $j < l$ such that Y_j and Y_l differ in exactly one position, d say, and, for some $0 \le i < r-2$, there exist a pair of words $S_i = [X_i, \overline{X}_i]$ and $S_{i+1} = [X_{i+1}, \overline{X}_{i+1}]$ such that X_i and X_{i+1} also differ in exactly position d. We claim that the words

$$[Y_j, X_i + Y_j, \overline{Y}_j, X_i + \overline{Y}_j]$$

and

$$[Y_j, X_{i+1} + Y_j, \overline{Y}_j, X_{i+1} + \overline{Y}_j]$$

still lie adjacent in the main list. For if not, then some list $\mathcal{S}(Y_m)$ $(m \neq l)$ must have been inserted between them. This only occurs if Y_j and Y_m differ in exactly position d. This in turn implies that $Y_l = Y_m$—a contradiction, since these words are distinct. Therefore, we can insert a cyclic shift of the code $\mathcal{S}(Y_l)$ between the words

$$[Y_j, X_i + Y_j, \overline{Y}_j, X_i + \overline{Y}_j]$$

and

$$[Y_j, X_{i+1} + Y_j, \overline{Y}_j, X_{i+1} + \overline{Y}_j]$$

using Lemma 18, extending the main list.

Executing this process beginning with $\mathcal{S}(Y_1)$ and ending with $\mathcal{S}(Y_{2^{n-1}-1})$, we obtain a list of all $2^{n-1}r$ inequivalent self-dual words which obviously satisfy Property C1.

Observe that in the above procedure, we never insert any words in positions between the last two words and the first word of the initial list $\mathcal{S}(Y_0)$. These three words are

$$[0^{2n-4}10001^{2n-4}0111]$$
$$[0^{2n-4}10011^{2n-4}0110]$$
$$[0^{2n-4}00011^{2n-4}1110].$$

Thus these words remain the last two words and first word of the final list, so that the final list satisfies Property C3.

Examining the last list inserted, we see that Lemma 18 guarantees that there are pairs of consecutive words in the list which differ in positions n up to $2n-1$. Moreover, from the choice of words $Y_0, \cdots, Y_n$ and Lemma 18, there are pairs of consecutive words in the list which differ in positions 0 up to $n-1$. Hence Property C2 holds. □

An immediate consequence of Example 4 and Lemma 19 is the following theorem:

Theorem 20: For every $m \ge 3$, there exists an arrangement of the self-dual words of length $2n = 2^{m+1}$ satisfying Properties C1 to C3.

For $m \ge 3$ and $n = 2^m$, let the list of words in Theorem 20 be $S_0, S_1, \cdots, S_{r-1}$, where of course $r = 2^n/2n$. Consider the list $S_0, S_1, \cdots, S_{r-2}$. Now for each $0 \le i < r-2$, S_i and S_{i+1} differ in exactly two positions, while ES_{r-2} differs in exactly two positions from S_0. Thus Theorem 15 applies (with $j = 2n-1$) to show:

Theorem 21: If n is a power of 2, $n \ge 8$, then there exists a single-track Gray code of length n and period $2^n - 2n$.

Next we will prove that the single-track Gray codes obtained in Theorem 21 are optimal if we use the construction given in this section, i.e., we will prove that there is no arrangement $S_0, S_1, \cdots, S_{r-1}$, $r = 2^n/2n$ such that

1) S_i and S_{i+1} (subscripts taken modulo r) differ in exactly two positions.
2) $E^j S_0$ differs in exactly two positions from S_{r-1}, for j relatively prime to $2n$.

It is known from [11] that there is a one-to-one mapping from the set of all self-dual words of length $2n$ to the set of all necklaces of length n and odd weight. This mapping, $D = E + 1$, maps a self-dual word

$$[s_0, s_1, \cdots, s_{n-1}, \overline{s}_0, \overline{s}_1, \cdots, \overline{s}_{n-1}]$$

to the odd weight necklace

$$[s_0 + s_1, s_1 + s_2, \cdots, s_{n-1} + s_0]$$

while its inverse D^{-1} maps an odd weight necklace $[t_0, t_1, \cdots, t_{n-1}]$ to the self-dual word

$$\left[0, t_0, t_0 + t_1, \cdots, \sum_{i=0}^{n-2} t_i, \right.$$
$$\left. 1, 1 + t_0, 1 + t_0 + t_1, \cdots, 1 + \sum_{i=0}^{n-2} t_i \right].$$

Using the mapping D and its inverse, we deduce that the required arrangement $S_0, S_1, \cdots, S_{r-1}$ is equivalent to an arrangement of all the odd weight necklaces $T_0, T_1, \cdots, T_{r-1}$ satisfying

1) T_i and T_{i+1} (subscripts taken modulo r) differ in exactly two adjacent positions.
2) $E^j T_0$ differs in exactly two adjacent positions from T_{r-1}, for some j relatively prime to $2n$.

Lemma 22: There is no arrangement of all the odd weight necklaces having Properties 1 and 2 above.

Proof: First note that T_i and T_{i+1} differ in one even position and one odd position and hence, since $r-1$ is odd, T_0 and T_{r-1} differ in an odd number of even positions and in an odd number of odd positions. The same is true of T_0 and $E^j T_0$ since j, being relatively prime to $2n$, is odd. Thus $E^j T_0$ and T_{r-1} differ in an even number of even positions and in an even number of odd positions. Therefore $E^j T_0$ and T_{r-1} cannot differ in exactly two consecutive positions. □

VI. Heuristic Methods

Our recursive methods for generating single-track Gray codes require seed-codes $S_0, S_1, \cdots, S_{r-1}$ of small lengths and having the various properties required in Constructions A and B. If the length n is such that the number of full-period necklaces

$$\frac{1}{n}\sum_{d|n}\mu(d)2^{n/d}$$

is small, then seeds can be found by hand, as in Example 2. But if the number of full-period necklaces is large we need the aid of the computer. We have developed a simple greedy algorithm which gives excellent results. First, we take from each necklace the cyclic shift which is the least as a binary number to be the representative of the necklace. Then we list all these representatives in lexicographic order from least to greatest. From the results of [12], there are always at least as many full-period odd-weight necklaces as full-period even-weight necklaces. We form a list $\mathcal{L}(n)$ by taking in the same order all the even-weight necklaces and an equal number of odd-weight necklaces. Again, from [12], there are

$$\frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)R(n,d)$$

full-period necklaces of even weight, where

$$R(n,d) = \begin{cases} 2^{d-1}, & \text{if } \frac{n}{d} \text{ is odd} \\ 2^d, & \text{if } \frac{n}{d} \text{ is even} \end{cases}$$

and so $\mathcal{L}(n)$ has size

$$\frac{2}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)R(n,d).$$

If we can rearrange $\mathcal{L}(n)$ to form a list satisfying the hypotheses of Theorem 4, then we can obtain a code of length n that has optimal period by this necklaces construction. Likewise, if we arrange $\mathcal{L}(n)$ to form lists satisfying the properties required in Constructions A or B, then we will have optimal seeds for those constructions.

To this end, we apply the following algorithm to $\mathcal{L}(n)$:

Algorithm A

Let S_0 be the word $[0^{n-1}1]$, which is the first word in $\mathcal{L}(n)$. Assume we have a sequence $S_0, S_1, \cdots, S_k$ such that S_i and S_{i+1} $(0 \le i < k)$ differ in exactly one position. We find S_{k+1} as follows:

1) Find the first word S in $\mathcal{L}(n)$ which is not one of the S_i's and such that S_k and $E^m S$ differ in exactly one position, for some m. Take $S_{k+1} = E^m S$.
2) If no such S exists, then find the largest $l < k-1$ such that S_l differs in one position from $E^m S_k$, for some m, and let

$$S'_0, S'_1, \cdots, S'_k = S_0, \cdots, S_l, E^m S_k, E^m S_{k-1}, \cdots, E^m S_{l+1}.$$

Return to step 1) with this sequence of necklaces to find S'_{k+1}.

TABLE I
Best Known Length n Single-Track Gray Codes
(First column: Length n. Second column: Number of necklaces in $\mathcal{L}(n)$. Third column: Period of resulting single-track Gray code.)

n	Number of Necklaces in $\mathcal{L}(n)$	Period of Resulting Code
9	56	504
10	96	960
11	186	2046
12	330	3960
13	630	8190
14	1152	16128
15	2182	32730

If Algorithm A terminates with a list of necklaces containing all the words of $\mathcal{L}(n)$, then we have a Gray code $S_0, S_1, \cdots, S_{r-1}$ where, in general, the weight of S_{r-1} is even but greater than 2. We apply a further step similar to step 2) above to obtain a cyclic code:

3) If the weight of S_{r-1} is greater than 2 then find the least l such that the weight of S_{l+1} is 2 less than the weight of S_{r-1} and S_l differs in one position from $E^m S_{r-1}$, for some m. Let

$$S'_0, S'_1, \cdots, S'_{r-1} = S_0, \cdots, S_l, E^m S_{r-1}, \cdots, E^m S_{l+1}$$

and continue step 3) with this sequence of necklaces.

If step 3) finishes successfully, then we have a cycle $S_0, S_1, \cdots, S_{r-1}$ with S_0 of weight 1 and S_{r-1} of weight 2. Obviously, there exist two integers m_1 and m_2 such that $E^{m_1}S_0$ and $E^{m_2}S_0$ differ in exactly one position from S_{r-1}. If either m_1 or m_2 is relatively prime to n then Theorem 4 can be applied to give a single-track Gray code that is optimal by construction from necklaces (cf. the bound in [9, Section VI]). Notice that we will have $S_0 = [0^{n-1}1]$ and $S_1 = [0^{n-2}11]$, so we could take $S_0 = [0^{n-2}10]$ to increase the chances that one of m_1 or m_2 is relatively prime to n; the same is true of many other adjacent pairs S_i, S_{i+1}.

We have applied this heuristic method starting from $n = 7$; as described, it is successful for $n = 7, 9, 10, 11, 13$. For other values of n we have introduced a nondeterministic element into Algorithm A by choosing l randomly in steps 2) and 3). This quickly lead to codes for $n = 8, 12, 14, 15$. We did not consider $n > 15$, but we believe that the results would be similar. Applying Theorem 4, we obtain the best known single-track Gray codes for $9 \le n \le 15$. The parameters of these codes are summarized in Table I. For $n = 9, 11, 13$, the codes obtained have the highest possible period allowed by Lemma 2.

The cycles of necklaces resulting from Algorithm A cannot be used immediately as seeds in Construction B, for we require codes with $S_0 = [10^{n-1}]$ and $S_{r-1} = [10^{n-2}1]$ (Property B3), and having a change in each position (Property B2). In fact, the sequences generated above for $n = 8, 9, 10, 12, 14, 15$ can be ordered to have these properties, while for $n = 7$ a seed was obtained in Section IV. For the other values of n, we make a small change in Algorithm A. We begin our sequence of words with

$$S_i = [0^{i+1}1^{n-i-1}], \quad 0 \le i \le n-2$$

and continue with steps 1) and 2). Now we need S_{r-1} to be of weight $n-2$, which is usually the case, but if not, we use the following step:

3′) If the weight of S_{r-1} is not $n-2$ then find the largest l such that the weight of S_{l+1} is 2 more than the weight of S_{r-1} and S_l differs in one position from $E^m S_{r-1}$, for some m. Let

$$S'_0, S'_1, \cdots, S'_{r-1} = S_0, \cdots, S_l E^m S_{r-1}, \cdots, E^m S_{l+1}$$

and continue step 3′) with this sequence of necklaces.

Note that if step 3′) finishes successfully, then we obtain a list $S_0, S_1, \cdots, S_{r-1}$ such that

$$S_i = [0^{i+1} 1^{n-i-1}], \quad 0 \le i \le n-2$$

and S_{r-2} has weight $n-2$. S_{r-1} has two zeroes and it is easily verified that the words $S_0, \cdots, S_{n-3}$ can each be cyclically shifted to obtain a new list

$$S'_0, \cdots, S'_{n-3}, S_{n-2}, \cdots, S_{r-1}$$

with S'_0 and S_{r-1} differing in exactly one position. The adapted algorithm works deterministically for $n = 11$ and with randomness for $n = 13$ to give lists which, when cycled, give optimal seeds with the properties required in Construction B.

Hence we have optimal seeds for Construction B for all $7 \le n \le 15$. These seeds can be used to obtain families of high-period single-track Gray codes for many lengths.

Similar ideas can be used to obtain seeds for Construction A, but the Property 2 required there seems to be quite restrictive and some effort is required to generate codes. A length 7 seed-code was given in Example 2. We give optimal length 9 and 10 seeds in the Appendix. We did not attempt to obtain a seed for $n = 8$ in view of the results of Section V.

VII. Conclusions and Open Problems

We have considered the problem of constructing high-period single-track Gray codes. We have presented new optimal codes for $n = 9, 11, 13$ and very good codes for $9 \le n \le 15$ (see Table I). In addition, we have given three iterative constructions for codes, along with some optimal seed-codes, and demonstrated how they can be used to obtain families of codes having very high periods.

In particular, we have obtained codes of length $n = 2^m$ and period $2^n - 2n$. From the conditions of Lemma 2, these codes are optimal if length $n = 2^m$ period 2^n single-track Gray codes do not exist: we conjecture that this is the case for $m \ge 2$. The conjecture is true for $m = 2$ [9] but is open even for $m = 3$.

We also conjecture that when n is a prime, all $\frac{2^n-2}{n}$ full-period necklaces of length n can be arranged to satisfy the hypotheses of Theorem 4. This is certainly true for the primes $3, 5, 7, 11$, and 13. A proof would lead to the construction of optimal single-track Gray codes for all prime lengths.

Finally, notice that all our constructions build codes of lengths a multiple of n from codes of length n. It would be convenient to have a method for producing good codes for all lengths. We leave this as an important open problem.

Appendix

We give a Gray code of 56 distinct length 9 full-period necklaces and a Gray code of 96 distinct length 10 full-period necklaces. These both have Properties A1 to A4.

Length 9:

[000000001]	[001010001]	[100101101]	[110111011]
[000000101]	[011010001]	[100100101]	[110101011]
[100000101]	[010010001]	[110100101]	[010101011]
[110000101]	[010110001]	[010100101]	[011101011]
[010000101]	[011110001]	[011100101]	[111101011]
[010000111]	[011111001]	[001100101]	[101101011]
[010000011]	[001111001]	[001100111]	[101101111]
[011000011]	[000111001]	[001101111]	[111101111]
[001000011]	[000111011]	[001101011]	[111001111]
[001000001]	[000110011]	[001101001]	[110001111]
[001100001]	[000110111]	[101101001]	[100001111]
[000100001]	[000110101]	[111101001]	[000001111]
[000110001]	[000111101]	[110101001]	[000000111]
[001110001]	[100111101]	[110111001]	[000000011]

Length 10:

[0000000001]	[0000110111]	[0100011011]	[0101010011]
[0000000101]	[0000110101]	[0100011111]	[0111010011]
[1000000101]	[0000111101]	[0110011111]	[0111110011]
[1100000101]	[0000111001]	[0010011111]	[0111111011]
[0100000101]	[0000111011]	[0010010111]	[0111101011]
[0100000111]	[1000111011]	[0010010011]	[0011101011]
[0100000011]	[1000101011]	[0010010001]	[0011101111]
[0110000011]	[1000101001]	[1010010001]	[1011101111]
[0010000011]	[1000101101]	[1010010011]	[1011100111]
[0010000001]	[1000100101]	[1010010111]	[1010100111]
[0011000001]	[1000110101]	[1010010101]	[1010100101]
[0001000001]	[1000110111]	[0010010101]	[1010101101]
[0001100001]	[1000110011]	[0110010101]	[1010101111]
[0011100001]	[1100110011]	[1110010101]	[1010101011]
[0010100001]	[1100110001]	[1110010111]	[1011101011]
[0110100001]	[1110110001]	[0110010111]	[1111101011]
[0100100001]	[1110010001]	[0110010011]	[1111101111]
[0101100001]	[0110010001]	[0110011011]	[1111001111]
[0111100001]	[0100010001]	[0110011001]	[1110001111]
[0111110001]	[0100010011]	[0110111001]	[1100001111]
[0011110001]	[0100010111]	[0100111001]	[1000001111]
[0001110001]	[0100010101]	[0100111011]	[0000001111]
[0000110001]	[0100011101]	[0100110011]	[0000000111]
[0000110011]	[0100011001]	[0101110011]	[0000000011]

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REFERENCES

[1] E. Gilbert, "Gray codes and paths on the n-cube," *Bell Syst. Tech. J.* vol. 37, pp. 815–826, 1958.

[2] J. R. Bitner, G. Ehrlich, and E. M. Reingold, "Efficient generation of the binary reflected Gray code and its applications," *Commun. ACM*, vol. 19, pp. 517–521, 1976.

[3] W. H. Mills, "Some complete cycles on the n-cube," *Proc. Amer. Math. Soc.*, vol. 14, pp. 640–643, 1963.

[4] T. A. Gulliver, V. K. Bhargava, and J. M. Stein, "On Gray codes and weight distributions," in *Proc. 16th Biennial Symp. on Communications* (Kingston, Ont., Canada, May 27–29, 1992).

[5] A. J. van Zanten, "Minimal-change order and separability in linear codes," *IEEE Trans. Inform. Theory*, vol. 39, pp. 1988–1989, 1993.

[6] A. Nijenhuis and H. S. Wilf, *Combinatorial Algorithms.* New York: Academic Press, 1975.

[7] E. M. Reingold, J. Nievergelt, and N. Deo, *Combinatorial Algorithms: Theory and Practice.* Englewood Cliffs, NJ: Prentice-Hall, 1977.

[8] V. Klee, "The use of circuit codes in analog-to-digital conversion," in *Graph Theory and its Applications*, B. Harris, Ed. New York: Academic Press, 1970.

[9] A. P. Hiltgen, K. G. Paterson, and M. Brandestini, "Single-track Gray codes," *IEEE Trans. Inform. Theory*, to appear.

[10] T. Etzion and A. Lempel, "Construction of de Bruijn sequences of minimum complexity," *IEEE Trans. Inform. Theory*, vol. IT-30, pp. 705–709, 1984.

[11] A. Lempel, "On a homomorphism of the de Bruijn graph and its application to the design of feedback shift registers," *IEEE Trans. Comput.*, vol. C-19, pp. 1204–1209, 1970.

[12] T. Etzion, "Self-dual sequences," *J. Combinatorial Theory, Ser. A*, vol. 44, pp. 288–298, 1987.